Solution Key to exam in Financial Econometrics A: Volatility Modelling, January 2016

Question A:

In finance (as well as elsewhere) much attention has recently been on OLS regression of returns y_t on predictive regressors x_t. We can write this as,

$$y_t = \beta x_t + \varepsilon_t, \qquad t = 1, 2, ..., T. \tag{1}$$

In the next we will make different assumptions about the stochastic properties of the innovations ε_t. The interest lies in testing if $\beta = 0$, which we write as $H : \beta = \beta_0 = 0$.

It is well-known that if ε_t are i.i.d.N$(0, 1)$ distributed the MLE of β is given by the OLS estimator,

$$\hat{\beta} = \sum_{t=1}^{T} y_t x_t \left(\sum_{t=1}^{T} x_t^2 \right)^{-1}. \tag{2}$$

Question A.1: When $\beta_0 = 0$ (that is, the hypothesis H holds) it follows that,

$$\hat{\beta} = \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t x_t \left(\frac{1}{T} \sum_{t=1}^{T} x_t^2 \right)^{-1} \tag{3}$$

Discuss conditions on x_t which ensure

$$\frac{1}{T} \sum_{t=1}^{T} x_t^2 \xrightarrow{p} \sigma_x^2 > 0. \tag{4}$$

Discuss briefly one or two examples of financial variables which may satisfy this condition. Explain why.

Solution: In line with the lecture notes, x_t should be weakly mixing with $E[x_t^2] < \infty$. Under this condition the convergence (in probability) would follow by a LLN for weakly mixing processes. One example is the case where x_t is a log-return following an ARCH(1) process with $\alpha < 1$. The good answer would also mention that one may use the drift criterion (under suitable conditions) to establish that x_t is weakly mixing with $E[x_t^2] < \infty$. No derivations are required.

Question A.2: Assume next that ε_t follows an ARCH specification, that is ε_t is an ARCH(1) process given by,

$$\varepsilon_t = \sigma_t \eta_t, \quad \eta_t \text{ i.i.d.N}(0, 1) \quad \text{and} \quad \sigma_t^2 = 1 + \alpha \varepsilon_{t-1}^2. \tag{5}$$

Under which conditions on α does it hold that

$$\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^4 \xrightarrow{p} \kappa_4 > 0? \tag{6}$$

Be specific about why it holds. (*Hint*: Recall that $E[\eta_t^4] = 3$)

Solution: Again, we need a LLN for weakly mixing processes. First, since η_t is Gaussian, ε_t is a Markov chain with a nice transition density. It can then be shown that ε_t satisfies a drift criterion with drift function $\delta(\varepsilon) = 1 + \varepsilon^4$ provided that $\alpha < 1/\sqrt{3}$. This ensures that ε_t is weakly mixing with $E\varepsilon_t^4 < \infty$.

Question A.3: Next assume that ε_t follow an ARCH specification and that $(\varepsilon_t, x_t)'$ is a Markov chain which satisfies a drift criterion with drift function,

$$\delta(\varepsilon, x) = 1 + \varepsilon^4 + x^4. \tag{7}$$

Show that under this assumption and that $E[\eta_t x_t|\varepsilon_{t-1}, x_{t-1}] = 0$,

$$\sqrt{T}\hat{\beta} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t x_t\left(\frac{1}{T}\sum_{t=1}^{T}x_t^2\right)^{-1} \xrightarrow{d} N\left(0, \sigma_{\varepsilon x}^2/\left(\sigma_x^2\right)^2\right) \tag{8}$$

where $\sigma_{\varepsilon x}^2 = E\left(\varepsilon_t^2 x_t^2\right)$ is finite.

Solution: Since $(\varepsilon_t, x_t)'$ satisfies a drift criterion with drift function $\delta(\varepsilon, x) = 1 + \varepsilon^4 + x^4$, $(\varepsilon_t, x_t)'$ is weakly mixing with $E[\varepsilon_t^4] + E[x_t^4] < \infty$. By the LLN for weakly mixing processes,

$$\frac{1}{T}\sum_{t=1}^{T}x_t^2 \xrightarrow{p} \sigma_x^2.$$

Next, we establish that $T^{-1/2}\sum_{t=1}^{T}\varepsilon_t x_t \xrightarrow{d} N\left(0, \sigma_{\varepsilon x}^2\right)$ using a CLT for martingale differences from the lecture notes. Since (ε_t, x_t) is weakly mixing it suffices to show that $E[\varepsilon_t x_t|\varepsilon_{t-1}, x_{t-1}] = 0$ and $E[(\varepsilon_t x_t)^2] < \infty$. The latter condition is ensured by the Hölder (or Cauchy-Schwarz) inequality: $\sigma_{\varepsilon x}^2 = E[(\varepsilon_t x_t)^2] \leq (E[\varepsilon_t^4]E[x_t^4])^{1/2} < \infty$. We have that

$$\begin{aligned} E[\varepsilon_t x_t|\varepsilon_{t-1}, x_{t-1}] &= E[\sigma_t\eta_t x_t|\varepsilon_{t-1}, x_{t-1}] \\ &= \sigma_t E[\eta_t x_t|\varepsilon_{t-1}, x_{t-1}] \\ &= 0, \end{aligned}$$

where the latter equality follows by the assumption that $E[\eta_t x_t|\varepsilon_{t-1}, x_{t-1}] = 0$. We conclude that $T^{-1/2}\sum_{t=1}^{T}\varepsilon_t x_t \xrightarrow{d} N\left(0, \sigma_{\varepsilon x}^2\right)$. This combined with the result that $\frac{1}{T}\sum_{t=1}^{T}x_t^2 \xrightarrow{p} \sigma_x^2$ yields the desired result.

Question A.4: By Question A.3 one can conclude that despite ARCH effects the estimator of the OLS estimator $\hat{\beta}$ is still asymptotically Gaussian. Explain how you would find the MLE of β when ε_t is given by the ARCH(1) model.

Solution: The log-likelihood function is (up to a constant) given by

$$\sum_{t=1}^{T} \left\{ -\frac{1}{2} \log[\sigma_t^2(\alpha, \beta)] - \frac{(y_t - \beta x_t)^2}{2\sigma_t^2(\alpha, \beta)} \right\},$$

where $\sigma_t^2(\alpha, \beta) = 1 + \alpha(y_{t-1} - \beta x_{t-1})^2$. The maximum likelihood estimate of (α, β) can be obtained by numerical maximization of the function.

Question B:

Suppose that the efficient log-price of a share of stock at time t is given by

$$P(t) = \sigma W(t), \quad t \in [0, T],$$

where $W(t)$ is a Brownian motion, $\sigma > 0$ is constant, and $T > 0$.

Question B.1: What is the distribution of $P(t)$?
With $t - 1 \geq 0$, let

$$r(t) = P(t) - P(t-1).$$

What is the mean and variance of $r(t)$?

Solution: Using the definition of a Brownian motion, $P(t) \sim N(0, \sigma^2 t)$. Likewise, $E[r(t)] = 0$ and $E[r^2(t)] = \sigma^2$.

Question B.2: Suppose that the price P is observed at $n+1$ equidistant points between time t and $t-1 \geq 0$, that is we observe $\{P(t_i) : i = 0, 1, ..., n\}$ where $t_i = t - 1 + i/n$.
One way to measure the volatility of $r(t)$ would be to compute the realized volatility given by

$$RV(t, n) = \sum_{i=1}^{n} \left(P(t_i) - P(t_{i-1})\right)^2.$$

Use that

$$P(t_i) - P(t_{i-1}) = \sigma(W(t_i) - W(t_{i-1}))$$

in order to find the probability limit of $RV(t, n)$ as $n \to \infty$. Be precise about the arguments used for deriving the probability limit.
Give an interpretation of letting $n \to \infty$.

Solution: By the definition of a Brownian motion and and of t_i, $P(t_i) - P(t_{i-1}) = \sigma n^{-1/2} \eta_i$ (in distribution), where $\eta_i \sim$i.i.d.$N(0, 1)$. Hence

$$RV(t, n) = \sigma^2 n^{-1} \sum_{i=1}^{n} \eta_i^2.$$

By the LLN for i.i.d. processes, $n^{-1} \sum_{i=1}^{n} \eta_i^2 \xrightarrow{p} 1$, and hence $RV(t, n) \xrightarrow{p} \sigma^2$. Alternatively, from the lecture notes we know that for a general class of continuous-time processes,

$$RV(t, n) \xrightarrow{p} [P](t) - [P](t-1) = \sigma^2 t - \sigma^2 (t-1) = \sigma^2 \quad \text{as } n \to \infty.$$

The "$n \to \infty$" can be interpreted as obtaining an increasing amount of observations in the time interval $[t-1, t]$, i.e. we increase the sampling frequency.

Question B.3: Suppose that we do not observe the efficient price $P(t)$, but instead we observe $\tilde{P}(t)$ which is $P(t)$ contaminated by some noise $\tilde{\varepsilon}(t)$, that is

$$\tilde{P}(t) = P(t) + \tilde{\varepsilon}(t), \quad t \in [0, T],$$

with

$$\tilde{\varepsilon}(t) = \tilde{\sigma}\tilde{W}(t),$$

where $\tilde{W}(t)$ is a Brownian motion and $\tilde{\sigma} > 0$ is constant.
Now the realized volatility measure $RV(t,n)$ from the previous question is infeasible due the fact that we do not observe $P(t)$. Instead we may compute

$$\widetilde{RV}(t,n) = \sum_{i=1}^{n} \left(\tilde{P}(t_i) - \tilde{P}(t_{i-1})\right)^2.$$

Assume that $W(t)$ and $\tilde{W}(t)$ are independent, that is $(W(t) : t \in [0,T])$ and $(\tilde{W}(t) : t \in [0,T])$ are independent. Similar to the previous question, derive the probability limit of $\widetilde{RV}(t,n)$ as $n \to \infty$. Compare with the probability limit of $RV(t,n)$.

Solution: Similar to the previous question,

$$\tilde{P}(t_i) - \tilde{P}(t_{i-1}) = P(t_i) - P(t_{i-1}) + \tilde{\varepsilon}(t_i) - \tilde{\varepsilon}(t_{i-1}) = n^{-1/2}\sigma\eta_i + n^{-1/2}\tilde{\sigma}\tilde{\eta}_i,$$

(in distribution) where $\tilde{\eta}_i \sim$i.i.d.$N(0,1)$. Since $(W(t) : t \in [0,T])$ and $(\tilde{W}(t) : t \in [0,T])$ are independent, η_i and $\tilde{\eta}_i$ are independent. Hence, by the LLN for i.i.d. processes,

$$\widetilde{RV}(t,n) = \sum_{i=1}^{n}(n^{-1}\sigma^2\eta_i^2 + n^{-1}\tilde{\sigma}^2\tilde{\eta}_i^2 - n^{-1}2\sigma\tilde{\sigma}\eta_i\tilde{\eta}_i) \xrightarrow{p} \sigma^2 + \tilde{\sigma}^2,$$

where it is used that $E(\eta_i\tilde{\eta}_i) = 0$.

Question B.4: Figure 1 contains a plot of the realized volatility of the return of the Euro/Dollar exchange rate. At day $t = 1, 2, ..., 796$ the realized volatility is based on $n = 47$ intra-daily return observations. Based on the figure and in light of your findings in the previous questions, do you think that the model $P(t) = \sigma W(t)$, from Question B.1 is suitable for the log-price of the exchange rate? Discuss briefly.

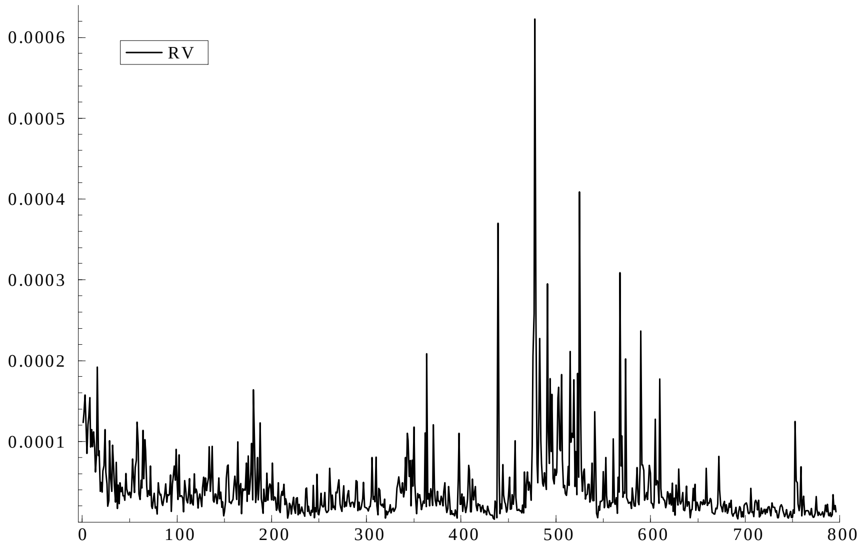

Figure 1: RV of Euro/Dollar returns

Solution: It is hard to say whether the model is suitable. Given that the model is correct, for n large, the realized volatility should be constant over time, according to the previous questions. This does not seem to be the case based on the graph. Note, however, that no uncertainty about the estimates are reported. Also, from Question B.2 $RV(t,n) = \sigma^2 n^{-1} \sum_{i=1}^{n} \eta_i^2 \sim \sigma^2 n^{-1} \chi_n^2$ for any fixed n. Hence for any fixed n, $RV(t,n)$ should be an i.i.d. chi-squared-type process. This does not seem to be a good approximation of the observed RV which seems to exhibit some degree of persistence.